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The Negative Binomial Distribution as a Renewal Model for the Recurrence of Large Earthquakes

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Abstract—The negative binomial distribution is presented as the waiting time distribution of a cyclic Markov model. This cycle simulates the seismic cycle in a fault. As an example, this model, which can describe recurrences with aperiodicities between 0 and 0.5, is used to fit the Parkfield, California earthquake series in the San Andreas Fault. The performance of the model in the forecasting is expressed in terms of error diagrams and compared with other recurrence models from literature.

Key words: Negative binomial distribution, renewal process, seismic cycle, earthquake forecasting.

1. Introduction

The *elastic-rebound model* is the canonical "macroscopic" theory of great earthquakes (Reid 1910; Scholz 2002). It states that a great earthquake will occur where large elastic strains have accumulated in the crust. The earthquake itself will relieve most of the strain, which will then accumulate slowly again by a steady input of tectonic stress until the elastic strain becomes sufficiently large for another earthquake to ensue. The duration of this *earthquake cycle* (the time between two consecutive large earthquakes) is the ratio of the strain released during an earthquake to the rate of input of tectonic strain by plate motion.

Because the Earth's crust is heterogeneous and faults are not isolated from each other, the earthquake cycle of a specific fault is not periodic. So, although the elastic-rebound model is in essence a deterministic theory, its application to a heterogeneous and interacting crust implies its translation into a probabilistic framework.

The variability of the duration of a cycle (either real earthquakes on a fault or synthetic earthquakes in a model) can be appropriately defined in the context of a probability density function (pdf) by means of the coefficient of variation, α, the ratio of the standard deviation σ to the mean μ of the pdf:

$$\alpha = \frac{\sigma}{\mu}. \tag{1}$$

In the seismological literature the coefficient of variation is also known as the *aperiodicity*, a very descriptive name when applied to the duration of the earthquake cycle: $\alpha = 0$ gives perfectly periodic cycles, $0 < \alpha < 1$ quasiperiodic cycles, and $\alpha > 1$ clustering of events. The case $\alpha = 1$ is particularly important because the exponential distribution has this property, and the exponential distribution is the pdf of an earthquake cycle where large earthquakes occur in time following a Poisson distribution (i.e., they are random in time). In actual seismic faults, the aperiodicity of the earthquake series is always <1 (Sykes and Menke 2006; Ellsworth *et al.* 1999; Abaimov *et al.* 2007).

Rikitake (1974) was the first to formally introduce a probabilistic description of the occurrence times of specific earthquakes. He treated earthquake recurrence as a *renewal process*, in which the times between successive events (in this case, the large earthquakes in a specific fault) are assumed to be independent and independently distributed random variables.

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Since then, several authors have proposed probabilistic versions of the elastic-rebound model in the shape of a plethora of probability distribution functions (pdfs) for the duration of the earthquake cycle: exponential (Utsu 1984; Sornette and Knopoff 1997; Matthews *et al*. 2002), Weibull (Utsu 1984; Sornette and Knopoff 1997; Matthews *et al*. 2002; Ferráes 2003; Gómez and Pacheco 2004; Yakovlev *et al.* 2006; Abaimov *et al*. 2007, 2008; Goltz *et al.* 2009), log-normal (Utsu 1984; Sornette and Knopoff 1997; Matthews *et al*. 2002; Gómez and Pacheco 2004; Ferráes 2005; Abaimov *et al*. 2007, 2008), gamma (Utsu 1984; Matthews *et al*. 2002; Gómez and Pacheco 2004; Ferráes 2005), power-law (Sornette and Knopoff 1997), Brownian passage time (Matthews *et al*. 2002; WGCEP 2003; Michael 2005; Yakovlev *et al.* 2006; Abaimov *et al*. 2007; Zöller *et al*. 2008), among others. However, due to the scarcity of registered large earthquakes in a specific fault (usually 4–10 earthquakes), the statistics upon which the selection of a specific pdf is based are poor. This means that different pdfs can fit the empirical distribution function.

Most of the probability distributions have been used solely for their statistical properties, with no relationships with the physics of the underlying process (elastic rebound theory). However, a subset of them has a physical rationale and from this point of view can be considered as better motivated. One example is the Brownian passage time distribution (BPT; Matthews *et al*. 2002), where the seismic cycle in a fault is modeled by the time evolution of the so-called Brownian relaxation oscillator.

Also, the majority of the probability distributions used in the context of earthquake recurrence are continuous. However, in the last 10 years, several *discrete* probability distributions that are the outcome of cellular automata models have been proposed (Vázquez-Prada *et al*. 2002; González *et al*. 2005; Tejedor *et al*. 2009). These discrete, cellular automata-based probability distributions share with the BPT distribution their physical motivation, as the models behind these discrete probability distributions try to reproduce in a few cellular automata rules the physics of a seismic fault under the elastic rebound assumption.

The aim of this paper is to present a discrete probability distribution, the negative binomial distribution (NBD) for the recurrence of large earthquakes. The study of one-way Markov cycles was presented in Tejedor *et al*. (2012), together with two of its limits, the so-called box model and the NBD. Here we focus on the NBD for its particular importance: the NBD seems to be the unique distribution that derives from the dynamics of a cellular automaton and simultaneously appears in general textbooks in probability and statistics. In Sect. 2, the NBD and its first moments are introduced. Section 3 recalls that the NBD is a special case of a waiting time distribution for a one-way Markov cycle, as deduced in Tejedor *et al*. (2012); Sect. 4 then uses this distribution as a renewal model for large earthquakes, using the earthquake series of the Parkfield segment of the San Andreas Fault as an example. The quality of the fit of the NBD to the empirical distribution function of the Parkfield series is compared to other renewal models used in the literature. Section 5 assesses the forecasting capabilities of the NBD by means of a reference prediction strategy and error diagrams. Finally, in Sect. 6, the most important conclusions drawn from the paper are stated. The computation of the asymptotic limit of the hazard rate for the NBD is detailed in the Appendix.

2. *The Negative Binomial Distribution*

As there are some different modalities of defining the NBD, we will now specify the form used in this paper.

A *negative binomial experiment* is a statistical experiment that has the following properties: The experiment consists of n repeated trials. Each trial can result in just two outcomes, a success or a failure. The probability of success, denoted by $1 - a$ $(a < 1)$, is the same on every trial. In consequence, the probability of failure is a. The trials are independent. And the experiment continues until N successes are observed. N is specified in advance.

The *negative binomial random variable* is the number n of repeated trials to produce N successes in a negative binomial experiment. The probability distribution of the negative binomial random variable is called an NBD. Its form is:

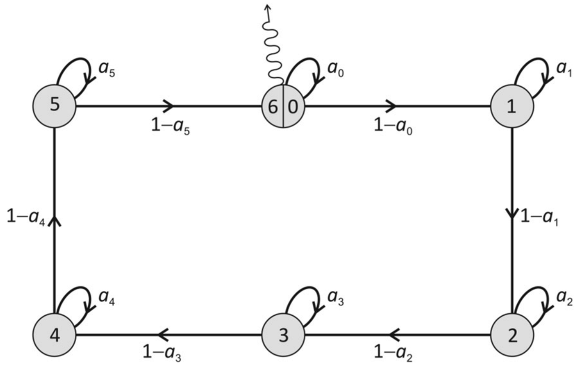

Figure 1
Scheme of a one-way Markov cycle with $N = 6$. The probability of staying in state i is a_i and the probability of jumping from state i to state $i + 1$ is $(1 - a_i)$. Jumping from one state to the next means that the fault has accumulated more strain energy. The *wavy line* between states *6* and *0* indicates that at the end of the cycle, all the stored energy is released

$$P_{N,a}(n) = (1-a)^N a^{n-N} \binom{n-1}{N-1} \tag{2}$$

The mean, variance, and coefficient of variation—or aperiodicity—of this distribution are:

$$\mu = \frac{N}{1-a}, \tag{3}$$

$$\sigma^2 = \frac{Na}{(1-a)^2}, \tag{4}$$

and

$$\alpha \equiv \frac{\sigma}{\mu} = \sqrt{\frac{a}{N}}, \tag{5}$$

respectively.

3. *The NBD as the Waiting Time Distribution in an Specific Markov Cycle Model*

Let us suppose a Markov chain with N sites forming a closed loop that has gone over clockwise (see Fig. 1 for illustration). The N sites are ordered by the index i, $i = 0, 1, \ldots, N-1, N$. As a genuine cellular automaton, time increases in discrete steps. At the beginning of each cycle, our system occupies the first position, $i = 0$. In the first time step, it makes a trial to pass to site $i = 1$. The probability of success is $(1 - a_0)$ and that of failure is a_0. Typically, after some trials, the system will occupy site 1. Now all is identical to the first case, except that the probability of passing from site 1 to site 2 is $(1 - a_1)$. Then the turn of sites is $2, 3, \ldots, N-1$.

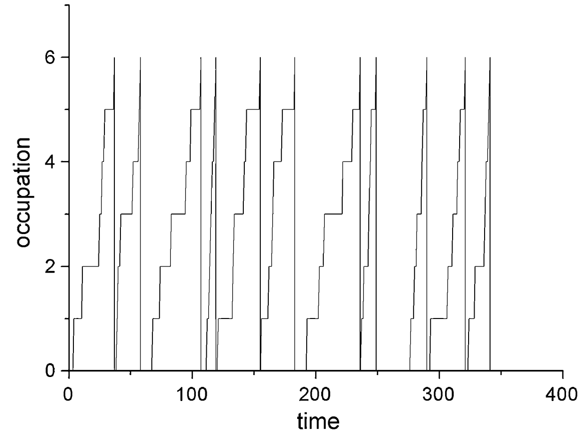

Figure 2
Occupation (number of occupied sites) of an $N = 6$ system as a function of time for 11 consecutive cycles. Note the repetitive pattern but the lack of perfect periodicity

When site N is occupied, the cycle ends. The system automatically passes to site 0 and a new cycle starts. Figure 2 shows an example of this process of slow filling and abrupt emptying for 11 consecutive cycles for a system with $N = 6$ and $(1 - a_i) = 1/N = 1/6$, for all states i.

The traveling in successive discrete steps around the cycle can be interpreted as a process of gradual increase of strain in fault, and thus this Markov cycle represents the seismic cycle in a fault. Site 0 represents the state with no strain and site N represents the state of maximum strain that is automatically released to pass to site 0. This sudden release of strain simulates the occurrence of a characteristic earthquake in the fault. Thus, in this model, a decrease in the strain, such as could take place in a random walk type model, is forbidden. This model is illustrated in Fig. 1 and materialized in the following Markov matrix:

$$[M] = \begin{pmatrix} a_0 & 1-a_0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 1-a_1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 1-a_2 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 1-a_3 & 0 \\ 0 & 0 & 0 & 0 & a_4 & 1-a_4 \\ 1-a_5 & 0 & 0 & 0 & 0 & a_5 \end{pmatrix} \tag{6}$$

Note that the number of parameters in this discrete model is $N + 1$: the length of the cycle, N, plus the value of the N parameters a_i. Using standard techniques of Markov chains (Tejedor *et al.* 2012), one can obtain, in a closed form, the distribution function of the cycle lengths in this model:

$$P_N(n) = \prod_{i=0}^{N-1} (1 - a_i) \sum_{i=0}^{N-1} \left[\frac{a_i^{n-1}}{\prod_{j(\neq i)=0}^{N-1} (a_i - a_j)} \right], \quad n = N, N+1, \ldots, \infty \tag{7}$$

It is clear that until time step $n = N$, the probability of completing a cycle is null. In seismology this is called a stress-shadow.

A property of this general model is that no matter what the value of its parameters are, the aperiodicity is lower than 1.

When the N parameters a_i are equal,

$$a = a_1 = a_2 = \cdots = a_N \tag{8}$$

Equation (7) becomes Eq. (2). That is, if Eq. (8) is fulfilled, an NBD is the waiting time distribution of the Markov cycle.

After this hypothesis, the pdf has only two parameters, N and a. This bi-parametric freedom can be used for fitting purposes, including, of course, the seismic cycles. In this paper, however, we will step forward with an additional simplification by relating them in the form:

$$1 - a = \frac{1}{N} \tag{9}$$

After this new hypothesis, there is only one free parameter and each cycle of the model can be intuitively associated with the *ordered* filling of a box with N positions. The new simplified NBD is

$$P_N(n) = \left(\frac{1}{N}\right)^N \left(\frac{N-1}{N}\right)^{n-N} \binom{n-1}{N-1}, \quad n = N, N+1, \ldots \infty \tag{10}$$

In the next section, we will see that $N = 6$ is the most appropriate size of the model to fit the recurrence of earthquakes in the Parkfield, California section of the San Andreas Fault. For this case, the pdf in Eq. (10) is simply:

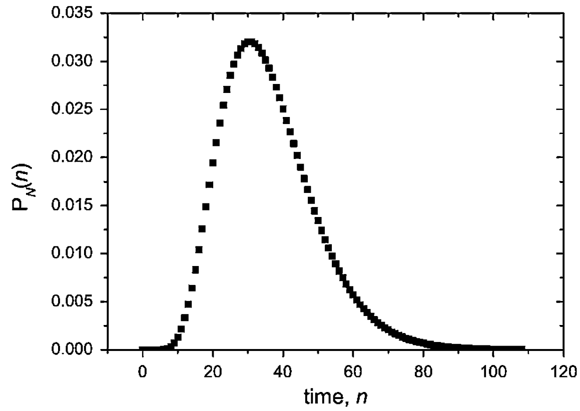

Figure 3
Probability density function of the NBD for the case $N = 6$, $(1 - a) = 1/6$

$$P_6(n) = \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{n-6} \binom{n-1}{5}, \quad n = 6, 7, \ldots \infty \tag{11}$$

The values of its mean ad aperiodicity are:

$$\mu_6 = 36 \quad \text{and} \quad \alpha_6 = 0.373 \tag{12}$$

Figure 3 plots the NBD written in Eq. (11). To mark the discrete nature of the probability distribution, only points for integer time steps have been drawn, with no line connecting them.

4. Applications of the NBD in Seismicity and Earthquake Forecasting: the Parkfield Series

Including the latest event, the Parkfield series (Bakun and Lindh 1985; Bakun 1988; Michael and Jones 1998) consists of seven $M_w \approx 6$ mainshocks, which occurred on 9 January 1857; 2 February 1881; 3 March 1901; 10 March 1922; 8 June 1934; 28 June 1966 and 28 September 2004. In consequence, the duration (in years) of the six observed inter-event times are: 24.07, 20.08, 21.02, 12.25, 32.05 and 38.25. The mean value μ_{Pk}, the sample standard deviation σ_{Pk} (the square root of the bias-corrected sample variance), and the aperiodicity α_{Pk} of this six-data series are:

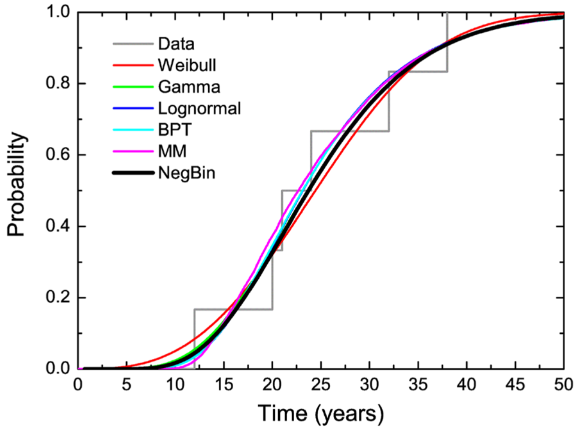

Figure 4
Fit of the NBD model (*black continuous line*) to the Parkfield series (*gray step-like line*) and comparison with other statistical models used in the literature

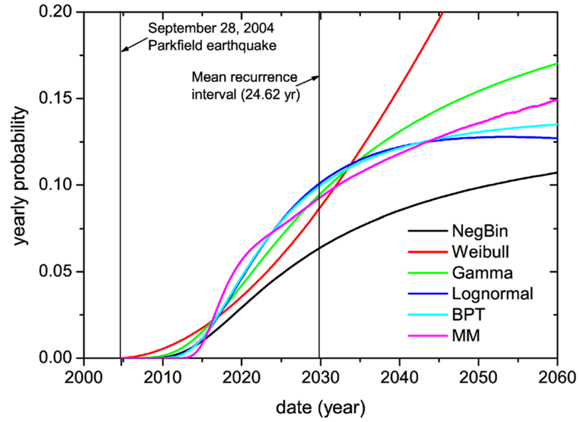

Figure 5
Yearly conditional probability for the Parkfield series as predicted by the negative binomial model compared to other statistical models used in the literature

$$\mu_{\mathrm{Pk}} = 24.62\,\text{years}; \quad \sigma_{\mathrm{Pk}} = 9.25\,\text{years}; \quad \alpha_{\mathrm{Pk}} = 0.3759 \tag{13}$$

Now, we will proceed to fit these data using the simplified NBD written in Eq. (11). Its aperiodicity is given by

$$\alpha_{\mathrm{NBD}} = \sqrt{\frac{N-1}{N^2}} \tag{14}$$

As we want a distribution with the same aperiodicity (and mean) as the Parkfield series, taking α_{Pk} from Eq. (13) and substituting it in Eq. (14), we have $N = 5.8$. But because N is a discrete quantity, we use the nearest integer, $N = 6$.

However, for fitting the data, it is necessary to assign a definite number of years to the non-dimensional time step of the model. This second parameter will be called τ. From Eqs. (3) and (9), we have that for the NBD, $\mu = N^2 = 36$ time steps. This mean cycle length (in non-dimensional time steps) should be equal to the mean recurrence time of the Parkfield series, $\mu_{\mathrm{Pk}} = 24.62$ years, so that $\tau = 0.68$ years per time step of the model. In Fig. 4, we have plotted the empirical distribution function of the Parkfield series (gray step-like line) and the fit to the cumulative NBD with $N = 6$ (black continuous line), together with five other (cumulative) distribution functions used as renewal models in the literature: Weibull, gamma, log-normal, BPT and minimalist model (MM; Vazquez-Prada *et al*. 2002). It is quite obvious from the figure that the performance of all six models is good and very similar, including the NBD. Indeed, the residuals for the NBD evaluated at the midpoints of the horizontal segments of the empirical distribution function are the lowest of the six tested models.

The NBD (and any of the other models shown in Fig. 4) can be used to estimate the time-dependent probability of having an earthquake as a function of the time elapsed since the last earthquake in the series (28 September 2004). This estimation can be carried out with the hazard rate function,

$$h_{N,a}(n) = \frac{P_{N,a}(n)}{\sum_{i=n}^{\infty} P_{N,a}(i)} \tag{15}$$

For discrete distributions like the NBD, the hazard rate is the probability for an earthquake to occur at time step n on the condition that it has not occurred until time step $n-1$. However, in the seismological literature, it is customary to express the likelihood of a future earthquake using the *yearly conditional probability* of earthquake occurrence, $P(n|\Delta t = 1\ \text{year})$, instead of the hazard rate. This function gives the probability of having an earthquake during the next year, provided it has not occurred before:

$$P_{N,a}(n|\Delta t = 1\,\text{year}) = \frac{S_{N,a}(n+\Delta t) - S_{N,a}(n)}{1 - S_{N,a}(n-1)} \tag{16}$$

where $S_{N,a}(n) = \sum_{n'=N}^{n} P_{N,a}(n')$ is the cumulative distribution function. The yearly conditional probability function for the Parkfield series is illustrated in

Fig. 5. Again, as in Fig. 4, the NBD and five other models are compared. The present yearly probability of earthquake occurrence is 0.004, i.e., there is a 0.4 % probability of having an earthquake in the following 12 months. Obviously this probability is low because the earthquake cycle is in its early stages. When the cycle is at its average duration, 24.62 years, the yearly probability of earthquake occurrence will be 6 %.

Both the hazard rate and the yearly conditional probability functions for the NBD reach a constant value for large times. Inserting Eq. (2) into Eq. (15), one obtains that, for long times,

$$\lim_{n\to\infty} h_{N,a}(n) = 1 - a \tag{17}$$

The derivation of this equation can be found in the Appendix. If Eq. (9) is used instead (i.e., the one-parameter simplification of the NBD), the asymptotic limit of the hazard function is equal to $1/N$.

5. Error Diagrams for the Parkfield Example

A hint of the predictability of the large relaxations in this type of model is given by the *aperiodicity* of their time series. The aperiodicity, as stated in Sect. 1, is a quantitative measure of the lack of regularity of a time series. As the aperiodicity of this model is always <1, the occurrence of the large events is a quasi-periodic phenomenon. A robust way to assess the predictability of a time series is by trying to forecast its events by declaring alarms at particular times.

The aim is to declare alarms before all the events in order not to miss any, but to declare them just before each event in order to minimize the total alarm time. Many strategies can be devised to declare the alarms, but there is a *reference strategy* to which all others can be compared (Newman and Turcotte 1992; Vázquez-Prada *et al*. 2002; Keilis-Borok and Soloviev 2003). This strategy consists of waiting a fixed time after each event (waiting time w), then setting the alarm, and maintaining it until the occurrence of the next event (Fig. 6). If the following event in the time series occurs before the alarm is raised, it is counted as a prediction error; if the following event in the time series occurs after the alarm is raised, it is counted as a prediction of success and the alarm is then cancelled.

The events that are to be predicted (large earthquakes) are the vertical red bars numbered correlatively. An alarm (vertical black lines with rounded top) is set a fixed time interval after each event (waiting time) and the prediction is labeled error (E) or success (S), depending on whether the alarm was off or on when the event occurred, respectively. The fraction of errors is the number of events not predicted (one in the example, the second event) divided by the total number of events (five events), i.e., $f_e = 0.2$; and the fraction of alarm time is the total alarm time (blue sections of the time line: 29 time units) divided by the total duration of the time series (86 time units), i.e., $f_a = 0.34$ in the example shown in the figure.

The fraction of errors f_e (number of missed events divided by the total number of events) and the fraction of alarm time f_a (total alarm time divided by the total duration of the time series) can be computed as a function of the above mentioned waiting time w, and the purpose is to find the optimum waiting time. This optimum waiting time depends on the relative importance that failing to predict an event has compared to keeping the alarm on. An objective function, called loss function, L,that incorporates this trade-off in each particular case can be defined. Here, we will use the simplest of them, $L = f_e + f_a$, where failure to predict an event and a longer alarm time are equally penalized.

Thus, the aim is to find the waiting time $w = w^*$ that minimizes $L(w)$. This minimum value is denoted by $L^* \equiv L(w^*)$. The best way to graphically display this is by means of an error diagram, where the fraction of alarm time f_a runs along the horizontal axis and the fraction of errors f_e runs along the vertical axis. Error diagrams were introduced in earthquake forecasting by Molchan (1997), who contributed to the optimization of the earthquake prediction strategies with rigorous mathematical analysis.

A good strategy of forecasting must produce both small f_e and f_a, because both the prediction failures and the alarms are costly. A random guessing strategy (randomly turning the alarm on and off) will yield $L = 1$, a result which can be easily understood. The

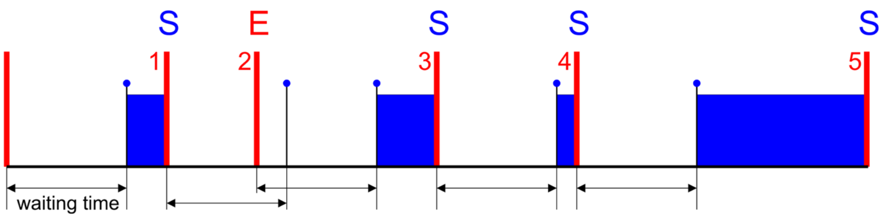

Figure 6
Reference strategy for the assessment of the predictability of a time series. *Red bars* are the earthquakes to be forecasted (five in the example). An *S* (success) above a *red bar* means that the earthquake has been successfully predicted, whereas an *E* (error) means that the earthquake has not been predicted. The *blue strips* stand for the time with the alarm on before each earthquake

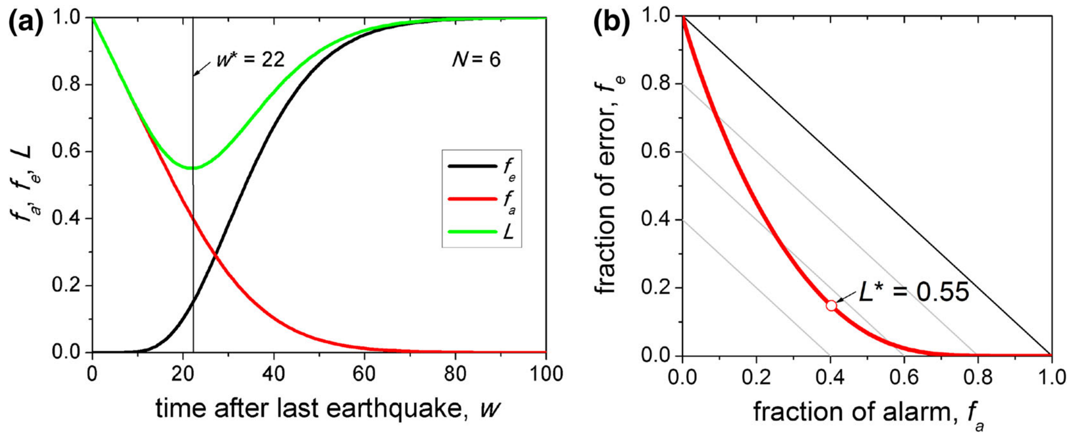

Figure 7
a Fraction of error f_e, fraction of alarm, f_a and loss function L as a function of the time after the last earthquake for a NBD model with $N = 6$. **b** *Error diagram* for the prediction strategy shown in **a**. The minimum value of the loss function is $L^* = 0.55$ for $w^* = 22$

alarm will be on, randomly, during a certain fraction of time, f_a. Thus, there will be a probability equal to f_a for it being on when an earthquake eventually occurs (and a probability of $1 - f_a$ for it being off). The result is that $f_e = 1 - f_a$. As a trivial special case, if the alarm is always on ($f_a = 1$), then all the earthquakes are "forecasted" ($f_e = 0$). Conversely, all the earthquakes are failures to predict if the alarm is always off. The random guessing strategy is considered as a baseline, so a forecasting procedure makes sense only if it gives $f_a + f_e < 1$.

Both functions, f_a and f_e, together with the loss function $L = f_e + f_a$ are plotted in Fig. 7a for the case $N = 6$, while Fig. 7b plots the error diagram for the same data. For each value of N, $L(w)$ has a minimum at a specific value of w, $w^*(N)$. As can be seen in Fig. 7, $w^*(6) = 22$, for which

$$f_a(w^*) = 0.403, \quad f_e(w^*) = 0.147, \quad L(w^*) = 0.550 \tag{18}$$

For the Parkfield sequence, w^* corresponds to

$$\tau w* = 15.0\,\text{years}$$

If the distribution derived from the NBD model correctly describes the recurrence of large earthquakes at Parkfield, an alarm connected 15 years after the last earthquake (beginning of the cycle) and disconnected just after the occurrence of each shock would yield the results given in Eq. (18). Note that this time is approximately equal to the difference between the mean and the standard deviation. This is reasonable because $w^* = 15$ years would capture most of the probability curve, as can be seen in Fig. 3.

6. Conclusions

We have introduced the NBD as a renewal model to describe the recurrence of large earthquakes in faults.

As a test ground of application, we have used the Parkfield series. The yearly conditional probability and other functions as predicted by the NBD are compared to other statistical models used in the literature, and a simple forecasting strategy has been evaluated using error diagrams.

Our results show that the NBD is competitive against other models, but general conclusions cannot be drawn because of the smallness of the sample.

The NBD seems to be the unique discrete distribution coming from a cellular automaton whose properties can be found in textbooks of probability and statistics.

In this paper, we have reduced one parameter of the distribution by relating the probability of advancing in the Markov process to the total number of steps in the cyclic chain. With this simplification, this model can be intuitively understood as the progressive ordered filling of a finite box.

Appendix: Asymptotic Behavior of the Hazard Rate Function

Recall that the N-step Markov-cycle distribution, Eq. (7), collapses to a NBD when all transition probabilities are equal, $a = a_1 = a_2 = \cdots = a_N$:

$$\begin{aligned} P_{N,a}(n) &= (1-a)^N a^{n-N} \binom{n-1}{N-1} \\ &= \left(\frac{1-a}{a}\right)^N a^n \frac{(n-1)\ldots(n-N+1)}{(N-1)!}. \end{aligned} \tag{19}$$

Using the definition of hazard rate for a discrete distribution, Eq. (15) we can write

$$\begin{aligned} h_{N,a}(n) &= \frac{P_{N,a}(n)}{\sum_{i=n}^{\infty} P_{N,a}(i)} \\ &= \frac{a^n(n-1)\ldots(n-N+1)}{\sum_{i=n}^{\infty} a^i (i-1)\ldots(i-N+1)} \\ &= \frac{1}{\sum_{i=1}^{\infty} a^{i-n} \frac{i-1}{n-1} \cdots \frac{i-N+1}{n-N+1}}. \end{aligned} \tag{20}$$

To proceed further, we make the following change of variable:

$$i - n = m. \tag{21}$$

With this change of variable, the hazard rate of the general, two-parameter NBD, Eq. (20), can be written as

$$h_{N,a}^{-1} = \sum_{m=0}^{\infty} a^m \left(1 + \frac{m}{n-1}\right) \ldots \left(1 + \frac{m}{n-N+1}\right). \tag{22}$$

In the long-time limit, i.e., when n tends to infinity, we have

$$\begin{aligned} \lim_{n\to\infty} h_{N,a}^{-1} &= \sum_{m=0}^{\infty} a^m (1 \times 1 \times 1 \times \cdots \times 1) \\ &= \sum_{m=0}^{\infty} a^m = \frac{1}{1-a}. \end{aligned} \tag{23}$$

So, in the general, two-parameter NBD the asymptotic limit of the hazard rate is:

$$\lim_{n\to\infty} h_{N,a} = 1 - a. \tag{24}$$

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